SHORT NOTE

Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 12, NUMBER 4 (2009) ISSN 1311-0454

ON THE RELATIONSHIPS OF THE R-FUNCTION OF LORENZO AND HARTLEY WITH OTHER SPECIAL FUNCTIONS OF FRACTIONAL CALCULUS

U.K. Saha [1], L.K. Arora [2], A.K. Arora [3]

Abstract

The R-function and its generalization denoted as G-function, both introduced in papers by Lorenzo and Hartley (L-H), are important for the applications of the fractional calculus. In [4], L-H obtained various relationships of the R-function with the Mittag-Leffler function, Agarwal's function, Erdélyi's function, Robotnov and Hartley function, and Miller and Ross function. In the next paper [5], they developed relationships of the R-function with the error function, product of exponential function and the complementary error function, and Dawson's integral. In this note, we like to stress to these and other relationships of the Lorenzo-Hartley R- and G-functions with the special functions of fractional calculus, as well as with classical special functions.

Mathematics Subject Classification: 26A33, 33E20, 33E50

Key Words and Phrases: R- and G-functions of Lorenzo-Hartley, Fox's H-function, Wright's generalized hypergeometric function, Kummer function, error and complementary error function, Dawson's integral

1. Introduction

The R-function and its generalization denoted as the G-function (but *not* the Meijer's G-function) were introduced by Lorenzo and Hartley [4] by means of the following series representations:

$$R_{q,\nu}[a,c,t] = \sum_{n=0}^{\infty} \frac{a^n (t-c)^{(n+1)q-1-\nu}}{\Gamma\left((n+1)q-\nu\right)}, \quad t > c \geq 0, q \geq 0, \Re(q-\nu) > 0, \quad (1)$$

and

$$G_{\alpha,\beta,\gamma}[a,c,t] = \sum_{n=0}^{\infty} \frac{(\gamma)_n a^n (t-c)^{(n+\gamma)\alpha-\beta-1}}{n!\Gamma\left((n+\gamma)\alpha-\beta\right)}, \tag{2}$$

where $\Re(\alpha\gamma-\beta) > 0, \Re(s) > 0, \left|\frac{a}{s^\alpha}\right| < 1$, and $(\gamma)_n$ is the Pochhammer symbol

$$(\gamma)_n = \begin{cases} 1, & n=0 \\ \gamma(\gamma+1)\dots(\gamma+n-1), & n \in N. \end{cases} \tag{3}$$

The Laplace transform of the R-function is given by ([4, p.4, eq. (21)]:

$$L\left\{R_{q,\nu}[a,c,t];s\right\} = \frac{\exp(-cs)\, s^\nu}{s^q-a},\ c \geq 0, \Re(q-\nu) > 0, \Re(s) > 0,$$

and especially for $c=0$ (see [4, p.4, eq. 19]),

$$L\left\{R_{q,\nu}[a,0,t];s\right\} = \frac{s^\nu}{s^q-a},\ \Re(q-\nu) > 0, \Re(s) > 0. \tag{4}$$

Let us note the relation with the exponential function:

$$R_{1,0}(a,0,t) = \exp(at),$$

and with many other elementary functions (see [4]).

2. Relationships of the R-function with other special functions

From the definition (1) of the R-function, the following relationships with other special functions are seen:

(i) Mittag-Leffler (M-L) function (see in books [7], [3])

$$R_{\nu,\mu}[a,c,t] = (t-c)^{\nu-\mu-1} E_{\nu,\nu-\mu}[a(t-c)^\nu],\ R_{\nu,\mu}[a,0,t] = t^{\nu-\mu-1} E_{\nu,\nu-\mu}[at^\nu]. \tag{5}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function (with 2 indices, as introduced by Wiman [10]):

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)},\ \Re(\alpha) > 0, \Re(\beta) > 0. \tag{6}$$

Indeed, the Laplace transform of (6) is well known (books [7], [3]) and gives same as (4):

$$L\left\{t^{\nu-\mu-1} E_{\nu,\nu-\mu}[at^\nu];s\right\} = \frac{s^\nu}{s^q-a},\ \Re(s) > 0, a \in \mathbb{C}, |as^{-\nu}| < 1.$$

(ii) Wright function ([11],[12])

$$R_{\nu,\mu}[a,c,t] = (t-c)^{\nu-\mu-1}{}_1\psi_1\left[\begin{array}{ll}(1,1) & \\ & ;a(t-c)^{\nu} \\ (\nu-\mu,\nu) & \end{array}\right], \tag{7}$$

where ${}_1\psi_1(z)$ is special case of the Wright's generalized hypergeometric function ${}_p\psi_q(z)$ defined by (see [11], [12], [9],[3])

$${}_p\psi_q\left[\begin{array}{ll}(a_1,A_1),\ldots,(a_p,A_p) & \\ & ;z \\ (b_1,B_1),\ldots,(b_q,B_q) & \end{array}\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+A_jn)}{\prod_{j=1}^{q}\Gamma(b_j+B_jn)}\cdot\frac{z^n}{n!}. \tag{8}$$

Especially, for $c=0$ this relation is confirmed by the same Laplace transform $(s^{\mu}/s^{\nu}-a)$ as for the corresponding Wright function in (7).

(iii) Fox's H-function (and thus, also a $\overline{H}$-function, [2]):

$$R_{\nu,\mu}[a,c,t] = (t-c)^{\nu-\mu-1}H^{1,1}_{1,2}\left[-a(t-c)^{\nu}\left|\begin{array}{l}(0,1)\\(0,1),(1-\nu+\mu,\nu)\end{array}\right.\right], \tag{9}$$

where

$$H^{m,n}_{p,q}\left[z\left|\begin{array}{l}(a_j,e_j)_{1,p}\\(b_j,f_j)_{1,q}\end{array}\right.\right] = \frac{1}{2\pi i}\int_L \theta(s)z^s ds, \tag{10}$$

is Fox's H-function ([6], [9], [3]), in which

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-f_js)\prod_{j=1}^{n}\Gamma(1-a_j+e_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=n+1}^{p}\Gamma(a_j-e_js)}. \tag{11}$$

Then, again in view of definition (1), the following particular cases of the Mittag-Leffler, Wright and Fox H-functions can be presented also in terms of the Lorenzo-Hartley R-functions, as:

(iv) Incomplete gamma function (see [8, p.439, eq. 45:6:2])

$$\gamma(\nu;x) = \Gamma(\nu)R_{1,0}(-1,0,x)R_{1,-\nu}(1,0,x), \tag{12}$$

where $\gamma(\nu;x)$ is the incomplete gamma function

$$\gamma(\nu;x) = \frac{\exp(-x)}{\nu}\sum_{n=0}^{\infty}\frac{x^{n+\nu}}{(\nu+1)_n}\,. \tag{13}$$

(v) Error function (see [8, p.387, eq. 40:6:2])

$$\operatorname{erf}(\sqrt{x}) = R_{1,0}(-1,0,x)R_{1,-\frac{1}{2}}(1,0,x), \text{ (see [5], eq. (A-2)).} \tag{14}$$

(vi) Complementary error function (see [8, p.390, eq. 40:12:5])

$$\operatorname{erfc}(x) = 1 - R_{1,0}(-1,0,x^2)R_{1,-\frac{1}{2}}(1,0,x^2), \tag{15}$$

(vii) Kummer function (see [8, p.464, eq. 47:6:1])

$$xM(1;\frac{3}{2};x^2) = \frac{\sqrt{\pi}}{2}R_{1,-\frac{1}{2}}(1,0,x^2), \tag{16}$$

where $M(a;c;x)$ is the Kummer function

$$M(a;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{x^n}{n!}. \tag{17}$$

As a matter of fact, the Kummer function has the same Laplace transform $\left(s^{-1/2}/s-1\right)$, as the corresponding R-function in (16).

(viii) Product of exponential function and complementary error function

$$\exp(x^2)\operatorname{erfc}(\pm x) = R_{\frac{1}{2},-\frac{1}{2}}(\mp 1,0,x^2). \tag{18}$$

This relation follows by comparing the definition (1) and the power series expansion for $\exp(x)\operatorname{erfc}(\pm\sqrt{x})$ as given in [8, p. 398, eq. 41:6:1]). See in Lorenzo and Hartley [5, p.21, eq.(A-1)]

(ix) Dawson's integral (see [8, p. 407, eq. 42:6:2])

$$\operatorname{daw}(x) = \frac{\sqrt{\pi}}{2}x\sum_{n=0}^{\infty}\frac{(-x^2)^n}{\Gamma(n+\frac{3}{2})} = \frac{\sqrt{\pi}}{2}R_{1,-\frac{1}{2}}(-1,0,x^2), \tag{19}$$

see also Lorenzo and Hartley [5, p.21, eq. (A-3)].

3. Relationships of the G-function with other special functions

Similarly, in view of its definition (2), the Lorenzo-Hartley G-function satisfies the following relationships with other special functions:

(i) Kummer function

$$xM(1;\frac{3}{2};x^2) = \frac{\sqrt{\pi}}{2}G_{1,-\frac{1}{2},1}(1,0,x^2). \tag{20}$$

(ii) Product of exponential function and complementary error function

$$\exp(x)\operatorname{erfc}(\pm\sqrt{x}) = \sum_{n=0}^{\infty}\frac{(\mp\sqrt{x})^n}{\Gamma(1+\frac{n}{2})} = G_{\frac{1}{2},-\frac{1}{2},1}(\mp 1,0,x), \tag{21}$$

$$\exp(x^2)\,\mathrm{erfc}(\pm x) = \sum_{n=0}^{\infty} \frac{(\mp x)^n}{\Gamma(1+\frac{n}{2})} = G_{\frac{1}{2},-\frac{1}{2},1}(\mp 1, 0, x^2). \tag{22}$$

(iii) Dawson's integrals

$$\mathrm{daw}(x) = \frac{\sqrt{\pi}}{2} x \sum_{n=0}^{\infty} \frac{(-x^2)^n}{\Gamma(n+\frac{3}{2})} = \frac{\sqrt{\pi}}{2} G_{1,-\frac{1}{2},1}(-1, 0, x^2), \tag{23}$$

$$\mathrm{daw}(\sqrt{x}) = \frac{\sqrt{\pi x}}{2} \sum_{n=0}^{\infty} \frac{(-x)^n}{\Gamma(n+\frac{3}{2})} = \frac{\sqrt{\pi}}{2} G_{1,-\frac{1}{2},1}(-1, 0, x). \tag{24}$$

For other relationships of the G-function (2) with various special functions, we refer the paper by Chaurasia and Pandey [1].

Editorial Note. The authors have submitted to the *FCAA* journal a longer manuscript, providing more details and proofs of the relationships. However, the Editor has attracted their attention about the fact that *the R- function is a special case of the Mittag-Leffler (M-L) function* and thus, belongs to the important class of the so-called Special Functions of Fractional Calculus (SF of FC). This notion gained recently much popularity in view of the use of the SF of FC as solutions of fractional order differential equations with applications to modeling real phenomena. So, we proposed to authors to reduce their submission to a short note. *Its publication aims only to emphasize one more interesting example of the M-L function and of the SF of FC*, and also to popularize the numerous results by Lorenzo and Hartley [4],[5] for the special functions R and G, applied as solutions of problems from applied science. For example, the role of the R-function is demonstrated in the dynamic thermocouple problem where it enables the analyst to directly inverse a transform in the Laplace domain solution (the operational s-form) so to obtain the time domain solution. The further generalization of the R-function, called as G-function, brings role in the effects of repeated and partially repeated fractional poles. This generalization carries increased time domain complexity, see details in [4].

References

[1] V.B.L. Chaurasia and S.C. Pandey, On the new computable solution of the generalized fractional kinetic equations involving the generalized function for the fractional calculus and related functions. *Astrophys. Space Sci.* **317** (2008), 213-219.

[2] A.A. Inayat-Hussain, New properties of hypergeometric series derivable from Feynman integrals: II. A generalization of the H-function. *J. Phys. A: Math. Gen.* **20** (1987), 4119-4128.

[3] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations.* Elsevier, North Holland Math. Studies 204, Amsterdam, etc. (2006).

[4] Carl F. Lorenzo and Tom T. Hartley, *Generalized Functions for the Fractional Calculus.* NASA/TP-1999-209424/REV1 (1999), 17 p.; Available electronically at http://gltrs.grc.nasa.gov/reports/1999/TP-1999-209424-REV1.pdf

[5] Carl F. Lorenzo and Tom T. Hartley, *R-Function Relationships for Application in the Fractional Calculus.* NASA/TM-2000-210361 (2000), 22 p.; Available electronically at http://gltrs.grc.nasa.gov/reports/2000/TM-2000-210361.pdf

[6] A.M. Mathai and R.K. Saxena, *The H-function with Applications in Statistics and Other Disciplines.* Wiley Eastern Ltd, N. Delhi, Bangalore, Bombay (1978).

[7] I. Podlubny, *Fractional Differential Equations.* Acad. Press, San Diego - N. York etc. (1999).

[8] J. Spanier and K.B. Oldham, *An Atlas of Functions.* Hemisphere Publishing Corporation, Springer-Verlag (1987).

[9] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-function of One and Two Variables with Application.* South Asian Publ., N. Delhi (1982).

[10] A. Wiman, Ueber den fundamentalsatz in der theorie der funktionen $E_\alpha(x)$. *Acta Mathematica* **29** (1905), 191-201.

[11] E.M. Wright, The asymptotic expansion of the generalized hypergeometric functions. *J. Lond. Math. Soc.* **10** (1935), 286-293.

[12] E.M. Wright, The asymptotic expansion of the generalized hypergeometric functions. *J. Lond. Math. Soc.* **46(2)** (1940), 389-408.

[1,2] *Dept. of Mathematics*
North Eastern Regional Institute of Science and Technology
(Deemed University), Nirjuli - 791109
Itanagar, Arunachal Pradesh, INDIA
e-mail: [1] *uksahanerist@gmail.com* , [2] *lkarora_nerist@yahoo.com*

[3] *Dept. of Mathematics, Govt. College Ajmer*
Rajasthan, INDIA
e-mail: arun0007@gmail.com

Received: Sept. 20, 2009